



# A note on $k$ -generalized projections<sup>☆</sup>

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## Abstract

In this note, we investigate characterizations for  $k$ -generalized projections (i.e.,  $A^k = A^*$ ) on Hilbert spaces. The obtained results generalize those for generalized projections on Hilbert spaces in [Hong-Ke Du, Yuan Li, The spectral characterization of generalized projections, *Linear Algebra Appl.* 400 (2005) 313–318] and those for matrices in [J. Benítez, N. Thome, Characterizations and linear combinations of  $k$ -generalized projectors, *Linear Algebra Appl.* 410 (2005) 150–159].

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In [2], it was defined a *generalized projection* as a complex matrix  $A$  satisfying  $A^2 = A^*$ . This concept was extended in [3] for infinite-dimensional Hilbert spaces. For  $H$  a Hilbert space, we shall denote

$$\mathcal{B}(H) = \{A / A \text{ is linear and bounded operator, } A : H \rightarrow H\}.$$

If  $k$  is an integer greater than 1, we define a  $k$ -generalized projection as an element  $A$  of  $\mathcal{B}(H)$  such that  $A^k = A^*$ , where  $A^*$  is the adjoint operator of  $A$ . Moreover, the  $n \times n$  complex matrices such that  $A^k = A^*$  (where  $A^*$  denotes its conjugate transpose) were characterized in [1].

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We recall that  $A \in \mathcal{B}(H)$  is said to be *normal* if  $AA^* = A^*A$ , it is said to be *orthogonal projection* if  $A^2 = A = A^*$ , and  $A$  is called *k-potent* if  $A^k = A$ . In particular,  $A$  is a *projection* if  $A^2 = A$  and  $A$  is *tripotent* if  $A^3 = A$ . In addition, the spectrum of  $A$  will be denoted by  $\sigma(A)$ .

The main purpose of this note is to give characterizations of the  $k$ -generalized projections by using the spectral theorem for normal operators on Hilbert spaces (see [4]). We quote this theorem for the sake of completeness.

**Theorem 1** [4]. *Let  $H$  be a Hilbert space and  $A \in \mathcal{B}(H)$ . If  $A$  is normal then there exists a unique resolution of the identity  $E$  on the Borel subsets of  $\sigma(A)$  which satisfies*

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda),$$

where  $E(\lambda)$  denotes the spectral projection associated with the spectral point  $\lambda \in \sigma(A)$  and  $E(\lambda) = 0$  if  $\lambda \notin \sigma(A)$ .

The main result of this note is the following.

**Theorem 2.** *Let  $H$  be a Hilbert space and  $A \in \mathcal{B}(H)$ . Then the following statements are equivalent.*

- (a)  $A$  is a  $k$ -generalized projection.
- (b)  $A$  is normal and  $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$ , where  $\sqrt[k+1]{1}$  denotes the unity roots of order  $k+1$ .
- (c)  $A$  is normal and  $(k+2)$ -potent.

In this case, one has

$$A = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda E(\lambda), \quad (1)$$

where  $E(\lambda) = 0$  if  $\lambda \notin \sigma(A)$  and  $\bigoplus$  stands for the direct sum.

**Proof.** (a)  $\Rightarrow$  (b). Suppose that  $A^k = A^*$ . It is evident that  $AA^* = A^*A$ , i.e.,  $A$  is normal. Theorem 1 assures that

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda) \quad (2)$$

and then  $0 = A^k - A^* = \int_{\sigma(A)} (\lambda^k - \bar{\lambda}) \, dE(\lambda)$ , which implies  $\lambda^k - \bar{\lambda} = 0$  for all  $\lambda \in \sigma(A)$ . The roots of this equation are 0 and  $\sqrt[k+1]{1}$  since if  $\lambda = re^{i\theta}$ , with  $r > 0$  and  $-\pi \leq \theta < \pi$ , then we get  $r^k e^{ik\theta} = re^{-i\theta}$  and so  $r = 1$  and  $e^{i(k+1)\theta} = 1$ , i.e.,  $\lambda = e^{i\theta} \in \sqrt[k+1]{1}$ . From (2), it is clear that (1) holds.

(b)  $\Rightarrow$  (c). If  $A$  is normal and  $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$  then (1) is true from Theorem 1. Now, since  $\lambda^{k+2} = \lambda$  for all  $\lambda \in \sigma(A)$ ,

$$A^{k+2} = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda^{k+2} E(\lambda) = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda E(\lambda) = A.$$

(c)  $\Rightarrow$  (a). If  $A$  is normal, from Theorem 1 one has that

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda). \quad (3)$$

From  $A^{k+2} = A$  we get that

$$0 = A^{k+2} - A = \int_{\sigma(A)} (\lambda^{k+2} - \lambda) dE(\lambda).$$

Hence,  $\lambda^{k+2} - \lambda = 0$  for all  $\lambda \in \sigma(A)$ . Now, it is easy to deduce  $\lambda^k = \bar{\lambda}$  for all  $\lambda \in \sigma(A)$  and so, from (3) we obtain  $A^k = A^*$ .

This completes the proof.  $\square$

Theorem 2 in [3] and Theorem 2.1 in [1] can be obtained as corollaries of Theorem 2.

**Corollary 1.** *Let  $H$  be a Hilbert space and let  $A \in \mathcal{B}(H)$  be a  $k$ -generalized projection.*

- (I) *If  $\sigma(A) \subseteq \mathbb{R}$  and*
  - (a)  *$k$  is even then  $A$  is a projection.*
  - (b)  *$k$  is odd then  $A$  is a tripotent operator.*
- (II) *If  $\sigma(A) \subseteq i\mathbb{R}$  and*
  - (a)  *$k$  is a multiple of 4 then  $A^3 = -A$ .*
  - (b)  *$k$  is not a multiple of 4 then  $A = O$ .*

**Proof.** By Theorem 2 we know that  $A$  is normal and  $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$ .

- (I) By hypothesis,  $\sigma(A) \subseteq \{0\} \cup (\sqrt[k+1]{1} \cap \mathbb{R})$ . If  $k$  is even then  $\sigma(A) \subseteq \{0, 1\}$ , hence  $A^2 = A$ . If  $k$  is odd then  $\sigma(A) \subseteq \{-1, 0, 1\}$ , hence  $A^3 = A$ .
- (II) In this case,  $\sigma(A) \subseteq i\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1})$ . If  $k$  is a multiple of 4 then  $i\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1}) = \{0, i, -i\}$  and hence  $A^3 + A = O$ . If  $k$  is not a multiple of 4 then  $i\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1}) = \{0\}$  and hence  $A = O$ . This conclude the proof.  $\square$

It is well known that:  $A$  is normal and  $\sigma(A) \subseteq \mathbb{R}$  if and only if  $A = A^*$  (i.e.,  $A$  is *self-adjoint*). So, the hypothesis that “ $A$  is a  $k$ -generalized projection and  $\sigma(A) \subseteq \mathbb{R}$ ” is equivalent to “ $A$  is a  $k$ -generalized projection and  $A^* = A$ ”. Analogously, the hypothesis that “ $A$  is a  $k$ -generalized projection and  $\sigma(A) \subseteq i\mathbb{R}$ ” is equivalent to “ $A$  is a  $k$ -generalized projection and  $A^* = -A$ ” (i.e.,  $A$  is *skew self-adjoint*).

**Corollary 2.** *Let  $H$  be a Hilbert space and let  $A \in \mathcal{B}(H)$  be a  $k$ -generalized projection. The range of  $A$  (denoted by  $\mathcal{R}(A)$ ) is closed.*

**Proof.** Since  $A$  is a  $k$ -generalized projection, by Theorem 2 we get that  $A$  is normal and its spectrum is finite, so 0 is not a limited point of the spectrum of the normal operator  $A$ , then  $\mathcal{R}(A)$  is closed. This completes the proof.  $\square$

A similar result to Theorem 2 can be established for matrices and it generalizes Corollary 4 in [3].

**Corollary 3.** *Let  $H$  be a Hilbert space and let  $A \in \mathcal{B}(H)$  be a  $k$ -generalized projection. Then  $A^{k+1}$  is an orthogonal projection.*

**Proof.** From Theorem 2, we get  $A^{k+2} = A$  and then  $(A^{k+1})^2 = A^{k+2}A^k = AA^k = A^{k+1}$ . Moreover,  $A^{k+1}$  is an orthogonal projection because

$$(A^{k+1})^* - A^{k+1} = (A^k A)^* - A^k A = (A^* A)^* - A^* A = 0,$$

since  $A^* A$  is self-adjoint. This completes the proof.  $\square$

It is clear that Corollaries 2 and 3 generalize the results given in Corollary 3 in [3].

## References

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